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COMMENT

Schwartz's method and Goldstein's eigenvalue problem

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Abstract. We use Schwartz's method to study a relativistic problem whose mathematical structure is similar to Goldstein's eigenvalue problem, i.e. the problem of a zero energy Klein–Gordon particle in an external Coulomb field. We show that Schwartz's method fails to give a correct answer. Our result suggests that additional information about the eigenvalue spectrum for energies different from zero is needed before deciding whether Goldstein's cut-off independent solution discussed recently by Delbourgo and Prasad is physically meaningful.

As is well known (Goldstein 1953), the ladder approximation to the Bethe–Salpeter equation for a fermion–antifermion system exchanging massless bosons reduces to a simple form when zero energy eigenvalues are considered for the pseudoscalar sector of the interaction (Goldstein's problem). Delbourgo and Prasad (1977) recently showed that an adaptation of Schwartz's method (Schwartz 1976) for handling nearly singular non-relativistic potentials led to a solution of Goldstein's problem which was in agreement with previous investigations (Delbourgo *et al* 1967). Two difficulties arise in connection with this particular solution (Goldstein's solution). First, Goldstein's solution does not satisfy Mandelstam's criterion for acceptability (Mandelstam 1955, Higashijima and Nishimura 1976) so that, strictly speaking, Goldstein's problem has no cut-off independent solution, although an acceptable (cut-off dependent) solution is arbitrarily close to Goldstein's solution if we take a sufficiently large (but finite) cut-off momentum. A detailed study of the connection between Goldstein's solution and a cut-off dependent solution can be found in the paper by Higashijima and Nishimura (1976). A second difficulty comes from the complexity of the fermion–antifermion problem when the energy eigenvalue is not zero. This complexity prevents us from drawing any conclusion about the physical relevance of Goldstein's solution. By this we mean that there is no *a priori* reason why Goldstein's solution should be the zero energy limit of the (unknown) energy spectrum of the full fermion–antifermion wave equation. The purpose of this comment is to illustrate this remark with a simple relativistic problem for which the eigenvalue spectrum can be computed for *all* energies. Specifically, we shall apply Schwartz's method to find for which value of the coupling constant a Klein–Gordon particle in a Coulomb field has zero energy. As will be seen below, the differential equation to be solved is similar to Goldstein's equation. However, Schwartz's method *fails* to yield a correct answer.

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Consider a Klein–Gordon particle of mass m in an external Coulomb field of charge Ze (we use units such that $\hbar = c = 1$ and put the electron charge equal to $-e$). The radial Klein–Gordon (KG) equation describing a zero energy bound state for a particle of charge $-e$ is:

$$\left[-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{l(l+1)}{r^2}-\frac{Z^2e^4}{r^2}+m^2\right]R_l(r)=0. \tag{1}$$

The mathematical structure of equation (1) is similar to the radial equation of Goldstein's problem studied by Delbourgo and Prasad (1977). Let us now, with Schwartz (1976), introduce a short-distance modification of the Coulomb interaction and write:

$$\frac{Z^2e^4}{r^2}\left(\frac{r}{r_0}\right)^{\epsilon} \tag{2}$$

instead of Z^2e^4/r^2 in (1). Equation (1) now becomes identical with Schwartz's original equation (Schwartz's model I) provided we make the substitution:

$$-E \to +m^2 \qquad (E \text{ negative}). \tag{3}$$

We now use Schwartz's method to find:

$$4m^2r_0^2=\exp\left[-\frac{4}{\epsilon}\ln\left(\frac{l+\frac{1}{2}}{Ze^2}+\frac{c_n}{2}\frac{\epsilon^{2/3}}{Ze^2}(2l+1)^{1/3}\right)\right], \tag{4}$$

where the c_n are coefficients given by Schwartz (1976). For small ϵ, (4) yields:

$$4m^2r_0^2=\exp\left[\frac{2}{\epsilon}\ln\left(\frac{e^4Z^2}{(l+\frac{1}{2})^2}\right)\right]\exp\left(\frac{-4c_n}{\epsilon^{1/3}(2l+1)^{2/3}}\right). \tag{5}$$

Formula (5) (with $m^2 \to -E$) is given by Schwartz (1976). It can be obtained from (4) by writing:

$$\begin{aligned}\ln&\left(\frac{l+\frac{1}{2}}{Ze^2}+\frac{c_n}{2}\frac{\epsilon^{2/3}}{Ze^2}(2l+1)^{1/3}\right)\\ &=\ln\left(\frac{l+\frac{1}{2}}{Ze^2}\right)+\ln\left(1+\frac{c_n}{2}\frac{\epsilon^{2/3}(2l+1)^{1/3}}{l+\frac{1}{2}}\right)\\ &\simeq\frac{c_n}{2}\epsilon^{2/3}\frac{(2l+1)^{1/3}}{l+\frac{1}{2}}+\ln\left(\frac{l+\frac{1}{2}}{Ze^2}\right).\end{aligned}$$

From either formula (4) or (5), one finds that, in the limit $\epsilon \to 0$:

$$e^2Z=l+\tfrac{1}{2} \qquad \text{(zero energy bound state)}. \tag{6}$$

For instance, formula (5) yields

$$\frac{Ze^2}{l+\frac{1}{2}}=\exp[+\tfrac{1}{2}\epsilon\ln(4m^2r_0^2)]\exp[2c_n(\epsilon^{2/3}/Ze^2)(2l+1)^{1/3}].$$

Thus we see from (6) that, in complete analogy with Goldstein's problem, Schwartz's method selects the least singular solution to equation (1). The trouble is that (6) is well known to be wrong. Indeed, for $e^2Z = l + \frac{1}{2}$, the Klein–Gordon equation has a bound state solution of energy

$$E_{\mathrm{KG}} = m\left[1 + \left(\frac{l+\frac{1}{2}}{n-(l+\frac{1}{2})}\right)^2\right]^{-1/2} \tag{7}$$

with $n = l+1, l+2, \ldots$ (Baym 1969).

Thus, the mathematical method we used *failed* to yield the correct solution to our problem. We were only able to verify this, however, by knowing the eigenvalue spectrum for $E_{\mathrm{KG}} \neq 0$. As our equation (1) and Goldstein's equation have a similar structure, our result suggests that only a study of the Bethe–Salpeter fermion–antifermion system for total energy eigenvalues different from zero could decide whether Goldstein's solution is indeed physically meaningful.

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